6.2 Orthogonal Sets

A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal** set if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

Example 1. Determine which sets of vectors are orthogonal.

(1) $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}.$

$$\vec{u}_1 \cdot \vec{u}_3 = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} = -3 - 16 + 21 = 2 \neq 0$$

Thus the set is not an orthogonal set.

(2) $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

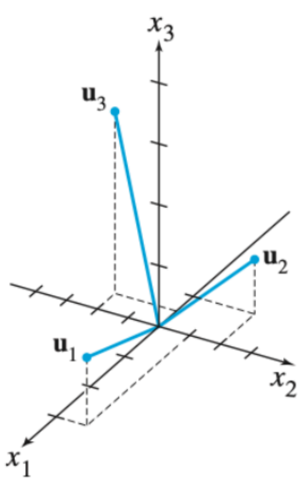


FIGURE 1

$$\vec{u}_1 \cdot \vec{u}_2 = 3 \times (-1) + 1 \times 2 + 1 \times 1 = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = 3 \times (-\tfrac{1}{2}) + 1 \times (-2) + 1 \times \tfrac{7}{2} = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = -1 \times (-\tfrac{1}{2}) + 2 \times (-2) + 1 \times \tfrac{7}{2} = 0$$

Since each pair of distinct vectors is orthogonal, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

proof on Page 359

Theorem 4 If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Definition. An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

The next theorem suggests why an orthogonal basis is much nicer than other bases. The weights in a linear combination can be computed easily.

Theorem 5. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \ldots, p)$$

Proof: $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p$

$\Rightarrow \vec{y}\cdot\vec{u}_1 = c_1\vec{u}_1\cdot\vec{u}_1 + c_2\underbrace{\cancel{\vec{u}_2\cdot\vec{u}_1}}_{0} + \cdots + c_p\underbrace{\cancel{\vec{u}_p\cdot\vec{u}_1}}_{0}$ since orthogonal

$\Rightarrow \vec{y}\cdot\vec{u}_1 = c_1\vec{u}_1\cdot\vec{u}_1$

$\Rightarrow c_1 = \dfrac{y\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}$

Similary, the result holds for every c_j $(j=1,\cdots,p)$

An Orthogonal Projection

Given a nonzero vector $\mathbf{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\mathbf{y}$ in $\mathbb{R}^n$ into the sum of two vectors, one a multiple of $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$.

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α and $\mathbf{z}$ is some vector orthogonal to $\mathbf{u}$. We can show that $\alpha = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}$:

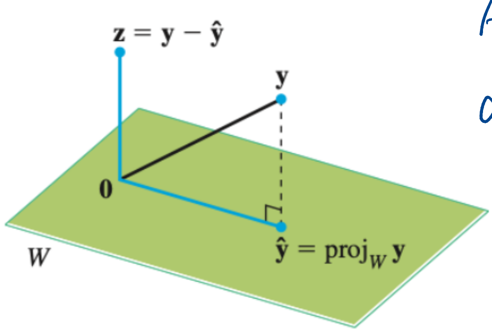


FIGURE 2

Finding α to make $\mathbf{y}-\hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

As $\hat{\vec{y}} = \alpha\vec{u}$, $\vec{z} = \vec{y} - \alpha\vec{u}$

also $\vec{z}$ is orthogonal to $\vec{u}$

$$0 = \vec{z}\cdot\vec{u} = (\vec{y} - \alpha\vec{u})\cdot\vec{u} = \underline{\vec{y}\cdot\vec{u} - \alpha\vec{u}\cdot\vec{u}}$$

$$\Rightarrow \alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$$

The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$,** and the vector $\mathbf{z}$ is called the **component of $\hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.** Let L be the line spanned by $\vec{u}$ (line through $\vec{u}$ and $\vec{0}$)

Note $\hat{\mathbf{y}}$ is also denoted by $\operatorname{proj}_L \mathbf{y}$ and is called **the orthogonal projection of $\hat{\mathbf{y}}$ onto** L.

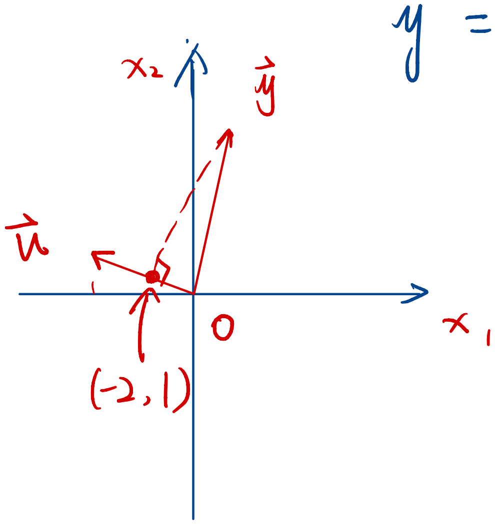
$$\hat{\mathbf{y}} = \operatorname{proj}_L \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

Example 2. Compute the orthogonal projection of $\underset{\vec{y}}{\begin{bmatrix}1\\7\end{bmatrix}}$ onto the line through $\underset{\vec{u}}{\begin{bmatrix}-4\\2\end{bmatrix}}$ and the origin.

$$\hat{\vec{y}} = \operatorname{proj}_L \vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \frac{\begin{bmatrix}1\\7\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix}}{\begin{bmatrix}-4\\2\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix}}\cdot\begin{bmatrix}-4\\2\end{bmatrix}$$

$$= \frac{-4+14}{16+4}\begin{bmatrix}-4\\2\end{bmatrix} = \frac{10}{20}\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$



Example 3. Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(1) Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.

(2) Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span $\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

(3) Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

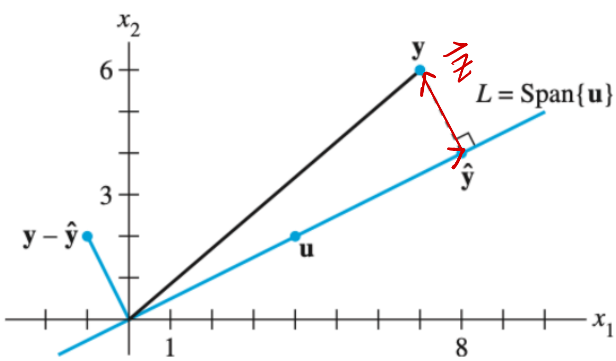


FIGURE 3 The orthogonal projection of $\mathbf{y}$ onto a line L through the origin.

ANS:

(1) $$\hat{\vec{y}} = \text{proj}_L \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$= \frac{\begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}} \vec{u}$$

$$= \frac{40}{20} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

(2) The component of $\vec{y}$ orthogonal to $\vec{u}$ is

$$\vec{z} = \vec{y} - \hat{\vec{y}} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Thus $$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$ $\Big(\vec{y} = \underbrace{\hat{\vec{y}}}_{\text{in span}\{\vec{u}\}} + \underbrace{\vec{z}}_{\text{orthogonal to } \vec{u}} \Big)$.

(3) Notice that the distance from $\vec{y}$ to L is the length of $\vec{y} - \hat{\vec{y}} = \vec{z}$.

$$\|\vec{y} - \hat{\vec{y}}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

Orthonormal Sets

A set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors. If W is the subspace spanned by such a set, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal basis** for W, since the set is automatically linearly independent, by Theorem 4.

Example 4. Determine which sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

(1) $$\underset{\overset{\shortparallel}{\vec{u}}}{\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}}, \underset{\overset{\shortparallel}{\vec{v}}}{\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}}$$

(2) $$\begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

(1) $\vec{u} \cdot \vec{v} = -\frac{2}{9} + \frac{2}{9} = 0$

Thus $\{\vec{u}, \vec{v}\}$ is an orthogonal set.

$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u} = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$

$\|\vec{v}\|^2 = \frac{1}{9} + \frac{4}{9} = \frac{5}{9} \neq 1$ and $\|\vec{v}\| = \frac{\sqrt{5}}{3}$

Thus $\{\vec{u}, \vec{v}\}$ is not an orthonormal set.

We can normalize $\vec{u}, \vec{v}$ to form the orthonormal set:

$$\left\{ \frac{\vec{u}}{\|\vec{u}\|}, \frac{\vec{v}}{\|\vec{v}\|} \right\} = \left\{ \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix} \right\}$$

(2) $\begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$$\underset{\vec{u}}{\parallel} \qquad \underset{\vec{v}}{\parallel} \qquad \underset{\vec{w}}{\parallel}$$

$$\vec{u} \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{w} = 0$$

$$\vec{v} \cdot \vec{w} = 0$$

Thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthogonal set.

$$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u} = \frac{1}{18} + \frac{16}{18} + \frac{1}{18} = 1$$

$$\|\vec{v}\|^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$\|\vec{w}\|^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

Thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is also an orthonormal basis.

Theorem 6. An $m \times n$ matrix U has orthonormal columns if and only if $\underset{n\times m}{U^T}\underset{m\times n}{U} = I_n$

Theorem 7. Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then
a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$ U preserves the length
b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ U preserves the inner product.
c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$ U preserves orthogonality

Exercise 5. Let $\mathbf{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

Solution. The distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin is $\|\mathbf{y} - \hat{\mathbf{y}}\|$. One computes that $\mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \begin{bmatrix} -3 \\ 9 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$, so $\|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{36 + 9} = 3\sqrt{5}$ is the desired distance.